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Optimal Power-Limited Rendezvous for Linearized Equations of Motion

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The form of solution of the optimal power-limited rendezvous problem for linear equations of motion is made to conform to previous developments by the author. The solution is then applied to the problem of rendezvous of a spacecraft with an object in the vicinity of a nominal Keplerian orbit. For the case in which the nominal orbit is circular, the controllability matrix can be inverted symbolically, resulting in a closed-form solution for the thrusting function. This result can be expressed as a closed-loop feedback law. For noncircular nominal orbits, a feedback law can be approximated by repeated numerical matrix inversions and reinitialization of the problem.

I. Introduction

OPTIMAL power-limited terminal rendezvous of a spacecraft based on linearized equations of motion offers the mathematical advantage of closed-form or nearly closed-form solutions of the linearized trajectory optimization problem. In many cases this leads to thrust control in the form of optimal feedback laws, or at least, control functions that can be easily reinitialized on a regular basis. This gain in adaptivity may, to some extent, compensate for the loss in accuracy due to the linearization.

There are many potential applications of a linearized analysis of optimal power-limited rendezvous. These typically involve the maneuvering of a low-thrust spacecraft near a real or fictitious object in a nominal orbit. Examples include maneuvers in the vicinity of artificial satellites, space stations, small comets, or asteroids. Examples where the object is fictitious include low-thrust satellite station keeping or interplanetary missions in which the usual high-thrust midcourse maneuver is replaced by a continuous long-term low-thrust correction toward the nominal orbital position and velocity.

Early work on the computation of power-limited, variable exhaust-velocity spacecraft trajectories was performed by Irving,[1] Ross and Leitman,[2] Edelbaum,[3] Saltzer and Fetheroff,[4] Melbourne,[5] and Melbourne and Sauer.[6,7] Immediately after these initial studies, investigators began to look at various linear models.[8–12] Billik[8] and Gobetz[9] used linearization about nominal circular orbits, Edelbaum[10] used linearization with respect to the orbital parameters of an ellipse, Gobetz[11] linearized about an ellipse of low eccentricity, and Euler[12] used the Tschauner-Hempel[13,14] linearization for general elliptical orbits.

More general discussions that include linear and nonlinear models for both power-limited and constant-exhaust-velocity problems can be found in the works of Edelbaum[15] and Marec.[16] Recently, optimal power-limited low-thrust trajectories to return to an original station in orbit after an impulsive maneuver away from this orbit were examined by Lembeck and Prussing[17] using equations linearized about a circular orbit. The same linearized equations were used by Coverstone-Carroll and Prussing[18] to investigate the cooperative power-limited rendezvous of two spacecraft near a nominal circular orbit. Power-limited trajectories in an inverse-square gravity field have also been considered by Prussing.[19]

In the present paper, we consider the fixed-time linear, power-limited rendezvous problem. We utilize a somewhat more general cost function than usual, an integral of the product of a weighting function and the square of the magnitude of the applied acceleration. A solution of this problem by Klamka[20] that is not based on control or optimization theory is used to construct a closed-loop feedback law. We condense some of the work slightly through the use of a transformation that avoids the necessity of inverting a fundamental matrix solution.[21]

This work is applied to the problem of rendezvous of a spacecraft with a body in Keplerian orbit and generalizes the work of Euler.[12] For the case of a circular orbit, the controllability matrix can be symbolically inverted, resulting in a completely closed-form solution of this optimal rendezvous problem.

II. Formulation and Solution for Linear Equations of Motion

We let Θ denote a closed, bounded interval of real numbers containing the independent variable θ, which represents time or true anomaly or other convenient measurement of an instant in the duration of a mission. The interval Θ is defined by its initial and terminal values $\theta_0 \leq \theta \leq \theta_f$ and we shall always assume that $\theta_f > \theta_0$. Using m and n to denote positive integers, we allow $A(\theta)$ and $B(\theta)$ to represent respective $n \times n$ and $n \times m$ matrices for each $\theta \in \Theta$. We also define γ as a positive continuous function on Θ called the weighting function. The generalized thrusting function $u\colon \Theta \to \mathcal{R}^m$ will be taken from a set $\mathcal{U}$ of admissible controls.

A. Statement of the Problem

The general optimal power-limited rendezvous problem for linear equations can be stated as follows.

Find a control function u from a set $\mathcal{U}$ of admissible controls that minimizes

$$J[u] = \int_{\theta_0}^{\theta_f} \gamma(\theta) u(\theta)^T u(\theta)\, d\theta \tag{1}$$

subject to the differential equation

$$y'(\theta) = A(\theta)y(\theta) + B(\theta)u(\theta) \tag{2}$$

the initial conditions

$$y(\theta_0) = y_0 \tag{3}$$

and the terminal conditions

$$y(\theta_f) = y_f \tag{4}$$

Here and throughout the superscript T refers to the transpose of a matrix or vector, the prime represents differentiation with respect to θ, and y_0 and y_f are elements of $\mathcal{R}^n$.

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B. The Solution

If the dynamical system (2) is controllable on Θ, then the optimal solution of this problem is known and can actually be found without Pontryagin's principle or calculus of variations. See Sec.1.7 of Klamka.[20] As in this reference, the set of admissible controls $\mathcal{U}$ is $L^2(\Theta, \mathcal{R}^m)$.

It follows from the controllability of the linear system (2) that the control

$$u(\theta) = \frac{1}{\gamma(\theta)} B(\theta)^T F(\theta_f, \theta)^T W(\theta_0, \theta_f)^{-1} [y_f - F(\theta_f, \theta_0) y_0] \quad (5)$$

establishes an optimal solution to this problem, where $F(\theta_f, \theta_0)$ denotes the state transition matrix associated with the homogeneous form of (2) and the $n \times n$ dimensional matrix

$$W(\theta_0, \theta_f) = \int_{\theta_0}^{\theta_f} \frac{1}{\gamma(\theta)} F(\theta_f, \theta) B(\theta) B(\theta)^T F(\theta_f, \theta)^T \,\mathrm{d}\theta \quad (6)$$

is invertible.[20] The optimal trajectory is given by

$$y(\theta) = F(\theta, \theta_0) y_0 + \int_{\theta_0}^{\theta} F(\theta, \tau) B(\tau) u(\tau) \,\mathrm{d}\tau \quad (7)$$

We shall present the optimal control (5) in an alternate form that we believe to be more convenient and that agrees with previous work.[21] We let $\Psi(\theta)$ denote a fundamental matrix solution of the adjoint system associated with Eq. (2), specifically

$$\Psi'(\theta) = -A(\theta)^T \Psi(\theta) \quad (8)$$

If we utilize the known relationship

$$F(\theta_f, \theta)^T = \Psi(\theta) \Psi(\theta_f)^{-1} \quad (9)$$

in Eqs. (5) and (6), then the optimal control takes the form

$$u(\theta) = [1/\gamma(\theta)] B(\theta)^T \Psi(\theta) M(\theta_0, \theta_f)^{-1} z_f \quad (10)$$

where the $n \times n$ dimensional matrix

$$M(\theta_0, \theta_f) = \int_{\theta_0}^{\theta_f} \frac{1}{\gamma(\theta)} \Psi(\theta)^T B(\theta) B(\theta)^T \Psi(\theta) \,\mathrm{d}\theta \quad (11)$$

is invertible and the vector

$$z_f = \Psi(\theta_f)^T y_f - \Psi(\theta_0)^T y_0 \quad (12)$$

has appeared previously.[21] We prefer the form of Eq. (10) over that of Eq. (5) because $\Psi(\theta)$, unlike $F(\theta_f, \theta)$, can be found without inverting a matrix, and because of the interpretation of z_f as the terminal value of the pseudostate vector

$$z(\theta) = \Psi(\theta)^T y(\theta) - \Psi(\theta_0)^T y_0 \quad (13)$$

previously defined.[21]

C. Reinitialization and Closed-Loop Control

Although we have a closed-form solution for this problem, it is based on the linearized equations (2). Various nonlinearities in an actual rendezvous maneuver may result in a drift from the value predicted by Eq. (7). This problem can be alleviated by updating the control function defined by Eq. (10) at specified intervals.

We can write Eq. (10) as

$$u(\theta) = -[1/\gamma(\theta)] B(\theta)^T \Psi(\theta) c \quad (14)$$

where $c = -M(\theta_0, \theta_f)^{-1} z_f$. At prespecified points $\theta_i \in \Theta$, we can update Eq. (14) by essentially starting over. Viewing the actual pseudostate $z(\theta_i)$ as a new initial condition, we replace Eq. (14) on the interval $[\theta_i, \theta_{i+1})$ by the adjusted control function

$$u(\theta) = -[1/\gamma(\theta)] B(\theta)^T \Psi(\theta) c_i \quad (15)$$

where

$$c_i = -M(\theta_i, \theta_f)^{-1} [z_f - z(\theta_i)] \quad (16)$$

Note that the matrix $M(\theta_i, \theta_f)^{-1}$ is easily determined a priori so that the only new information required for on-board computation is $z(\theta_i)$.

If the vector c is calculated instantaneously instead of at specified points, it can be viewed as a function of the pseudostate, and Eq. (16) is replaced by

$$c(\theta, z) = -M(\theta, \theta_f)^{-1} (z_f - z) \quad (17)$$

where Eq. (13) is replaced by

$$z = \Psi(\theta)^T y - \Psi(\theta_0)^T y_0 \quad (18)$$

and y is the actual state at the instant $\theta \in \Theta$. This enables us to replace Eq. (14) by an optimal closed-loop feedback law

$$u(\theta, z) = [1/\gamma(\theta)] B(\theta)^T \Psi(\theta) c(\theta, z) \quad (19)$$

Unfortunately this feedback law is singular at (θ_f, z_f) because $M(\theta_f, \theta_f)$ is the $n \times n$ zero matrix and is not invertible, so Eq. (17) is not applicable at θ_f. In practical problems, one usually needs only to reach some vicinity of a target point rather than the exact point. One can also "freeze" the values of $c(\theta, z)$ to a constant near the end of the flight interval.

III. Applications to Rendezvous near Keplerian Orbit

We now apply the preceding work to the problem of optimal power-limited rendezvous of a spacecraft with a satellite in Keplerian orbit. The equations of motion of the spacecraft can be found from several sources.[12–14,21–25] In these equations the first m rows of the matrix $B(\theta)$ are identically zero and the second m rows form a scalar function times an identity matrix. This causes the matrix $B(\theta)^T \Psi(\theta)$ to simplify to $\beta(\theta) R(\theta)^T$, where $R(\theta)^T$ is the $m \times n$ matrix that consists of the lower m rows of $\Psi(\theta)$[21,25] and $\beta(\theta)$ is a scalar. With this simplification, Eq. (11) becomes

$$M(\theta_0, \theta_f) = \int_{\theta_0}^{\theta_f} \frac{\beta(\theta)^2}{\gamma(\theta)} R(\theta) R(\theta)^T \,\mathrm{d}\theta \quad (20)$$

From this point on, we shall set $m = 3$ and $n = 6$.

A. Noncircular Orbits

This problem was first solved by Euler.[12]

We let θ denote the true anomaly of a satellite in Keplerian orbit. The Tschauner-Hempel[14,22] form of the equations of motion of the spacecraft relative to a rotating coordinate frame fixed in the satellite are

$$x_1''(\theta) = 2x_2'(\theta) + a_1(\theta)$$
$$x_2''(\theta) = \frac{3}{r(\theta)} x_2(\theta) - 2x_1'(\theta) + a_2(\theta) \quad (21)$$
$$x_3''(\theta) = -x_3(\theta) + a_3(\theta)$$

where

$$r(\theta) = 1 + e \cos\theta \quad (22)$$

and e denotes the eccentricity of the Keplerian orbit. [The term $r(\theta)$ is used to shorten the notation and should not be confused with the radial distance of the satellite from the mass that attracts it, although it is inversely proportional to this distance.] The transformed applied acceleration vector[22], $a(\theta) = (a_1(\theta), a_2(\theta), a_3(\theta))^T$ is given by

$$a(\theta) = \frac{k}{r(\theta)^3} u(\theta) \quad (23)$$

where $k = L^6/(\mu^4 m_0)$. This constant depends on L, the angular momentum of the satellite divided by its mass; μ, the universal gravitational constant times the mass of the central body of attraction; and m_0, the mass of the spacecraft. We assume that the applied acceleration is low, resulting in negligible change in the mass of the spacecraft over the interval Θ.

If we put Eqs. (21) in state-vector form we obtain Eq. (2). A comparison with Eq. (23) shows that

$$\beta(\theta) = \frac{k}{r(\theta)^3} \tag{24}$$

The coefficient matrix is given by

$$A(\theta) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3/r(\theta) & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{25}$$

This matrix depends on $r(\theta)$ and is therefore not constant unless the eccentricity of the nominal orbit is zero. It is known[22] that $r(\theta) > 0$ for each $\theta \in \Theta$, so β is nonzero on Θ. This linear system is controllable on Θ, and we can apply the preceding theory.

It was shown by some of the early researchers[1,2,15] in power-limited rendezvous and transfer over a time interval $[t_0, t_f]$ that the criterion to be minimized is the time integral of the square of the applied acceleration or of the applied thrust, that is, $J[u] = \int_{t_0}^{t_f} u(t)^T u(t)\,\mathrm{d}t$. From the law of conservation of angular momentum of a satellite in Keplerian orbit,[22] we have the relationship

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mu^2}{L^3}(1 + e\cos\theta)^2 \tag{26}$$

This enables us to change the variable from time to the true anomaly θ, where $\theta(t_0) = \theta_0$ and $\theta(t_f) = \theta_f$. With this change of variables, the cost function becomes

$$J[u] = \frac{L^3}{\mu^2}\int_{\theta_0}^{\theta_f} \frac{u(\theta)^T u(\theta)}{r(\theta)^2}\,\mathrm{d}\theta$$

It suffices to minimize any positive constant times this integral. We pick the positive constant to simplify Eq. (20) as much as possible. For this reason we choose the cost function

$$J[u] = k^2\int_{\theta_0}^{\theta_f} \frac{u(\theta)^T u(\theta)}{r(\theta)^2}\,\mathrm{d}\theta \tag{27}$$

Comparing with Eq. (1) we see that

$$\gamma(\theta) = \frac{k^2}{r(\theta)^2} \tag{28}$$

From Eq. (25) we can solve Eq. (8) and determine a fundamental matrix solution $\Psi(\theta)$. This has previously been found[21] for $e > 0$, and its bottom three rows determine the matrix

$$R(\theta)^T = \begin{bmatrix} 1 & -[1 + r(\theta)]\sin\theta & -r(\theta)^2 & -2r(\theta)^2 I(\theta) & 0 & 0 \\ 0 & -r(\theta)\cos\theta & \mathrm{e}r(\theta)\sin\theta & 2\mathrm{e}r(\theta)\sin\theta I(\theta) - \dfrac{\cos\theta}{r(\theta)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta \end{bmatrix} \tag{29}$$

where

$$I(\theta) = \int_{\theta_0}^{\theta} \frac{\cos\tau}{r(\tau)^3}\,\mathrm{d}\tau \tag{30}$$

The evaluation of this integral for various types of orbits has been determined.[24]

We now have enough information to specify the solution of the optimal Keplerian rendezvous problem. The optimal control function (10) becomes

$$u(\theta) = \frac{1}{kr(\theta)} R(\theta)^T M(\theta_0, \theta_f)^{-1} z_f \tag{31}$$

for each $\theta \in \Theta$. In this expression, we use numerical integration to evaluate the 6×6 matrix

$$M(\theta_0, \theta_f) = \int_{\theta_0}^{\theta_f} \frac{1}{r(\theta)^4} R(\theta) R(\theta)^T\,\mathrm{d}\theta \tag{32}$$

We then invert it numerically for use in Eq. (31). The problem is completely solved for $e > 0$ through the use of Eq. (29) in Eqs. (31) and (32). By inverting $M(\theta_i, \theta_f)$ a priori at various points $\theta_i \in \Theta$, we can repeatedly reinitialize the control function through Eqs. (15) and (16). In this manner we prepare, ahead of time, for drifting due to various inaccuracies in our linearized model.

B. Circular Orbits

If the nominal orbit is circular, that is, if $e = 0$, then we can say more about the optimal control function without resorting to numerical methods. We can, in fact, obtain the optimal control function in closed form.

The basic simplification is that Eq. (22) reduces to $r(\theta) = 1$, causing $\beta(\theta)$, $\gamma(\theta)$, and $A(\theta)$ to become constant. Solving the system (8) as before, we obtain a fundamental matrix solution $\Psi(\theta)$. From its bottom three rows, we obtain the matrix

$$R(\theta)^T = \begin{bmatrix} 3\theta & 1 & -2\sin\theta & 2\cos\theta & 0 & 0 \\ 2 & 0 & -\cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta \end{bmatrix} \tag{33}$$

From Eq. (24) we see that $\beta(\theta) = k$. Since $\gamma(\theta)$ is also a positive constant that can be taken arbitrarily, without changing the problem solution, we set $\gamma(\theta) = k^2$ in Eq. (1), simplifying Eq. (20) to

$$M(\theta_0, \theta_f) = \int_{\theta_0}^{\theta_f} R(\theta) R(\theta)^T\,\mathrm{d}\theta \tag{34}$$

Since $A(\theta)$ and $B(\theta)$ are constant, it can be shown that $M(\theta_0, \theta_f) = M(0, \theta_f - \theta_0)$. Letting $\delta = \theta_f - \theta_0$, the evaluation of the integral is a little more concise in terms of δ. We find a closed-form solution for $M(0, \delta)$, namely

$$\begin{bmatrix} 3\delta^3+4\delta & \frac{3}{2}\delta^2 & 6\delta\cos\delta-8\sin\delta & 6\delta\sin\delta+8\cos\delta-8 & 0 & 0 \\ \frac{3}{2}\delta^2 & \delta & 2\cos\delta-2 & 2\sin\delta & 0 & 0 \\ 6\delta\cos\delta-8\sin\delta & 2\cos\delta-2 & \frac{5}{2}\delta-\frac{3}{2}\sin\delta\cos\delta & -\frac{3}{2}\sin^2\delta & 0 & 0 \\ 6\delta\sin\delta+8\cos\delta-8 & 2\sin\delta & -\frac{3}{2}\sin^2\delta & \frac{5}{2}\delta+\frac{3}{2}\sin\delta\cos\delta & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\delta+\frac{1}{2}\sin\delta\cos\delta & \frac{1}{2}\cos^2\delta-\frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2}\cos^2\delta-\frac{1}{2} & \frac{1}{2}\delta-\frac{1}{2}\sin\delta\cos\delta \end{bmatrix} \quad (35)$$

The inverse of this matrix defines an optimal control function

$$u(\theta) = \frac{1}{k}R(\theta)^T M(0, \theta_f - \theta_0)^{-1} z_f \qquad (36)$$

The inverse of $M(0, \delta)$ can be found in closed form using a computer programmed for symbolic manipulation. This enables us to write an optimal closed-loop feedback law for power-limited rendezvous. If we reinitialize at the point (θ, z), Eq. (36) becomes

$$u(\theta, z) = \frac{1}{k}R(\theta)^T M(0, \theta_f - \theta)^{-1}(z_f - z) \qquad (37)$$

This closed-loop feedback law establishes an optimal transfer from any point $(\theta, z) \in \Theta \times \mathcal{R}^6$ to a point in the vicinity of the terminal point (θ_f, z_f).

We conclude by presenting a closed-form solution of $M(0, \delta)^{-1}$, found by a computer programmed for symbolic manipulation, that can be used in Eq. (36) or (37). If used in Eq. (37), we set $\delta = \theta_f - \theta$. The entries of $M(0, \delta)^{-1}$ are given by

$$n_{ij} = b_{ij}/d \qquad i = 1, 2, 3, 4$$

$$n_{ij} = 2c_{ij}/(\delta^2 - \cos\delta) \qquad i = 5, 6$$

where $b_{ij} = 0$ for $j > 4$ and $c_{ij} = 0$ for $i < 5$. We also have symmetry, that is, $n_{ij} = n_{ji}$ for each $i, j = 1, \ldots, 6$. For these reasons, it is sufficient to present the following:

$$\begin{aligned} d &= 16{,}384(1-\cos\delta)^2 + 14{,}592\sin\delta(\cos\delta-1)\delta \\ &+ 16(1-\cos\delta)(279\cos\delta-121)\delta^2 \\ &+ 32\sin\delta(111-18\cos\delta)\delta^3 \\ &+ (27\cos^2\delta-480\cos\delta-587)\delta^4 + 75\delta^6 \end{aligned}$$

$$\begin{aligned} b_{11} &= 192\sin\delta(1-\cos\delta) + 4(\cos\delta-1) \\ &\times (9\cos\delta+89)\delta + 100\delta^3 \end{aligned}$$

$$\begin{aligned} b_{12} &= 288\delta\sin\delta(\cos\delta-1) - 6\delta^2(\cos\delta-1) \\ &\times (9\cos\delta+89) - 150\delta^4 \end{aligned}$$

$$\begin{aligned} b_{13} &= 1024\sin\delta(\cos\delta-1) + 576\delta\sin^2\delta \\ &+ 8\delta^2\sin\delta(31-9\cos\delta) - 120\delta^3(\cos\delta+1) \end{aligned}$$

$$\begin{aligned} b_{14} &= -1024(1-\cos\delta)^2 + 576\delta\sin\delta(1-\cos\delta) \\ &+ 8\delta^2(\cos\delta-1)(9\cos\delta-31) - 120\delta^3\sin\delta \end{aligned}$$

$$\begin{aligned} b_{22} &= 3072\sin\delta(\cos\delta-1) - 80\delta(\cos\delta-1)(27\cos\delta-37) \\ &+ 96\delta^2\sin\delta(40-9\cos\delta) + 4\delta^3(27\cos^2\delta-287) + 300\delta^5 \end{aligned}$$

$$\begin{aligned} b_{23} &= -4096(1-\cos\delta)^2 + 4416\delta\sin\delta(1-\cos\delta) \\ &+ 160\delta^2(\cos\delta-1)(9\cos\delta+7) \\ &+ 24\delta^3\sin\delta(6\cos\delta-17) + 120\delta^4(2+\cos\delta) \end{aligned}$$

$$\begin{aligned} b_{24} &= 4096\sin\delta(1-\cos\delta) \\ &+ 192\delta(\cos\delta-1)(23\cos\delta+7) + 32\delta^2\sin\delta(45\cos\delta-19) \\ &- 48\delta^3(\cos\delta-1)(3\cos\delta-7) + 120\delta^4\sin\delta \end{aligned}$$

$$\begin{aligned} b_{33} &= -256\delta(\cos\delta-1)(3\cos\delta-5) \\ &+ 96\delta^2\sin\delta(8-7\cos\delta) + 32\delta^3(6\cos^2\delta-1) \\ &+ 18\delta^4\sin\delta\cos\delta + 30\delta^5 \end{aligned}$$

$$\begin{aligned} b_{34} &= 768\delta\sin\delta(1-\cos\delta) - 672\delta^2\sin^2\delta \\ &+ 96\delta^3\sin\delta(1+2\cos\delta) + 18\delta^4\sin^2\delta \end{aligned}$$

$$\begin{aligned} b_{44} &= 256\delta(\cos\delta-1)(3\cos\delta+5) + 96\delta^2\sin\delta(7\cos\delta+8) \\ &- 32(6\cos^2\delta+6\cos\delta+1) - 18\delta^4\sin\delta\cos\delta + 30\delta^5 \end{aligned}$$

$$c_{55} = \delta - \sin\delta\cos\delta$$

$$c_{56} = -\cos^2\delta$$

$$c_{66} = \delta + \sin\delta\cos\delta$$

IV. Conclusions

The solution of the optimal power-limited rendezvous problem for linear equations of motion can be put in an alternate form in order to utilize certain developments from previous work and applied to the problem of rendezvous of a spacecraft with a satellite in Keplerian orbit. For rendezvous with satellites in elliptical, parabolic, or hyperbolic orbits, the optimal solution is almost in closed form. All that is required is the numerical integration of the entries of a square matrix and numerical inversion of this matrix in order to define the optimal thrusting function. By performing this computational procedure repeatedly a priori, the solution can be made to adjust to deviations from the linearized model by reinitializing the thrusting function. For rendezvous with satellites in near-circular orbits, we can linearize about a nominal orbit that is circular and solve this linearized optimization problem in closed form. This solution was found through the use of a computer programmed for symbolic manipulation. In this manner we obtained the optimal power-limited thrusting function in closed form. Increased tolerance for inaccuracies in the model was gained by expressing this solution in the form of an optimal feedback law. Although this feedback law is singular at the final time, it can be used to direct the spacecraft until the final approach to the target.

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